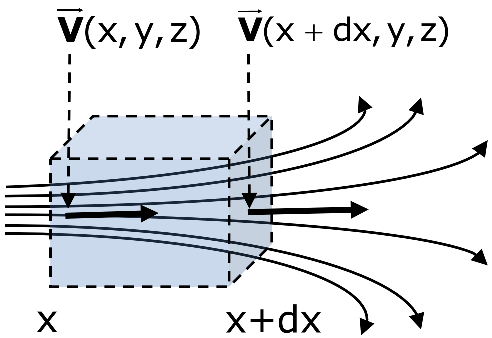
Lecture-4

Gradient of a scalar, divergence of a vector

- **Divergence**



1 – Definition

$\vec{\mathbf{v}}(x, y, z)$ is a differentiable vector field

$$\text{div } \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \partial_u v^u$$

2 – Physical meaning

div $\vec{\mathbf{v}}$ is associated to **local** conservation laws: for example, we'll show that if the mass of fluid (or of charge) outcoming from a domain is equal to the mass entering, then

$$\text{div } \vec{\mathbf{v}} = 0$$

$\vec{\mathbf{v}}$ is the fluid velocity (or the current) vectorfield

- **Laplacian: definitions**

1 – Scalar Laplacian. $\phi(x,y,z)$ is a differentiable scalar field

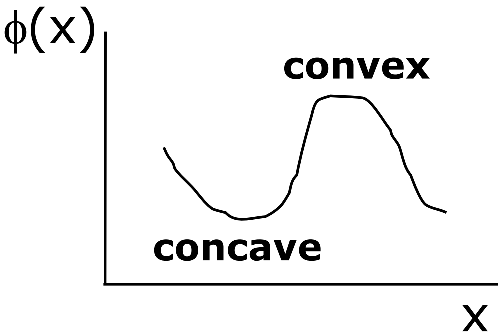
$$\Delta\phi = \nabla^2\phi = \text{div}(\mathbf{grad}\ \phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \partial_u\partial_u\phi$$

2 – Vector Laplacian. $\vec{\mathbf{v}}(x,y,z)$ is a differentiable vector field

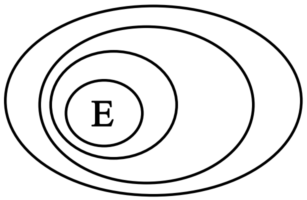
$$\begin{cases} \nabla^2 v_x = \dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_x}{\partial y^2} + \dfrac{\partial^2 v_x}{\partial z^2} \\ \nabla^2 v_y = \dfrac{\partial^2 v_y}{\partial x^2} + \dfrac{\partial^2 v_y}{\partial y^2} + \dfrac{\partial^2 v_y}{\partial z^2} \\ \nabla^2 v_z = \dfrac{\partial^2 v_z}{\partial x^2} + \dfrac{\partial^2 v_z}{\partial y^2} + \dfrac{\partial^2 v_z}{\partial z^2} \end{cases}$$

$$\Delta\vec{\mathbf{v}} = \Delta v_x \vec{\mathbf{u}}_x + \Delta v_y \vec{\mathbf{u}}_y + \Delta v_z \vec{\mathbf{u}}_z$$

- **Laplacian: physical meaning**



As a second derivative, the one-dimensional Laplacian operator is related to minima and maxima: when the second derivative is positive (negative), the curvature is concave (convexe).



In most of situations, the 2-dimensional Laplacian operator is also related to local minima and maxima. If v_E is positive:

$\Delta\phi = -v_E$: maximum in E (ϕ(E) > average value in the surrounding)

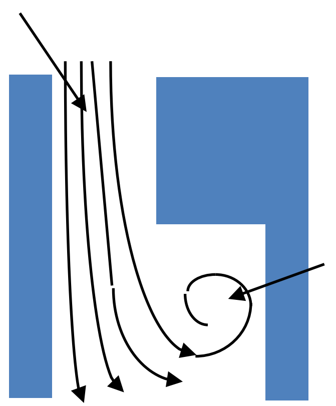
$\Delta\phi = v_E$: minimum in E (ϕ(E) < average value in the surrounding)

- # Curl

1 – Definition. $\vec{a}(x, y, z)$ is a differentiable vector field

$$\textbf{curl}\ \vec{\mathbf{A}} = \nabla \times \vec{\mathbf{A}} = \det\begin{bmatrix} \vec{\mathbf{u}}_x & \vec{\mathbf{u}}_y & \vec{\mathbf{u}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{bmatrix} = \epsilon_{ijk}\ \vec{\mathbf{u}}_{\mathbf{i}} \partial_j a_k$$

$$= \vec{\mathbf{u}}_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \vec{\mathbf{u}}_y \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \vec{\mathbf{u}}_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)$$



2 – Physical meaning: **curl** $\vec{\mathbf{v}}$ is related to the **local** rotation of the vectorfield:

If $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, **curl** $\vec{\mathbf{v}} = 2\vec{\boldsymbol{\omega}}$

$\vec{\mathbf{v}}$ is the fluid velocity vectorfield

Another form of the vector product :

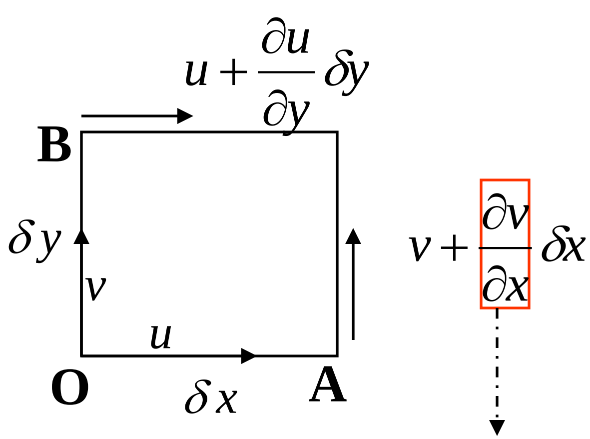
$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

is the "curl" of a vector ; $\nabla \times A = curl\ A$

What is its physical meaning?

Assume a two-dimensional fluid element

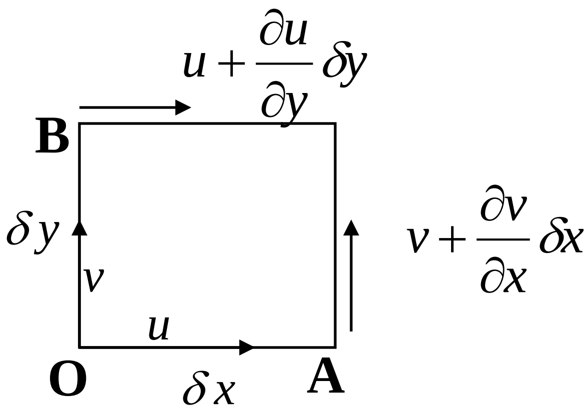


$$\nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Regarded as the angular velocity of OA, direction : **k**

Thus, the angular velocity of OA is $k\dfrac{\partial v}{\partial x}$; similarily, the angular velocity of OB is $-k\dfrac{\partial u}{\partial y}$

The angular velocity of the fluid element is the average of the two angular velocities :



$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

$\boldsymbol{\omega}$

$$\nabla \times \mathbf{u} = 2\omega\,\mathbf{k}$$

This value is called the "vorticity" of the fluid element, which is twice the angular velocity of the fluid element. This is the reason why it is called the "curl" operator.

$$\nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Coordinates other than cartesian

- Cylindrical polar coordinates (r, θ, z)
  - the edge of the increment element is general curved.
  - If **a, b, c** are unit vectors defined as point **P** :

$$\boldsymbol{\delta}\,\mathbf{r} = \delta r\,\mathbf{a} + r\,\delta\theta\,\mathbf{b} + \delta z\,\mathbf{c}$$

$$\mathbf{dr}\cdot\nabla = \mathrm{d} \qquad \mathbf{dr} = \boldsymbol{\delta}\,\mathbf{r} \to 0$$

$$\nabla = \frac{\partial}{\partial r}\mathbf{a} + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{b} + \frac{\partial}{\partial z}\mathbf{c}$$

The gradient of a scalar point function U :

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{a} + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{b} + \frac{\partial U}{\partial z}\mathbf{c}$$

Assuming that the vector **A** can be resolved into components in terms of **a**, **b**, and **c** :

$$\mathbf{A} = A_r\mathbf{a} + A_\theta\mathbf{b} + A_z\mathbf{c}$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}A_z$$

$$\nabla \times A = \left[\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right]\mathbf{a} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\mathbf{b} + \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\mathbf{c}$$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

The gradient of a scalar point function U :

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{a} + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{b} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\mathbf{c}$$

Assuming that the vector **A** can be resolved into components in terms of **a**, **b**, and **c** :

$$\mathbf{A} = A_r\mathbf{a} + A_\theta\mathbf{b} + A_\phi\mathbf{c}$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi$$

$$\nabla \times A = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{a} + \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial \phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right]\mathbf{b} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\mathbf{c}$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

3. Differential operators

- **Summary**

Operator	**grad**	**div**	**curl**	**Laplacian**
is	a vector	a scalar	a vector	a scalar (*resp.* a vector)
concerns	a scalar field	a vector field	a vector field	a scalar field (*resp.* a vector field)
Definition	$\nabla\phi$	$\nabla \cdot \vec{\mathbf{v}}$	$\nabla \times \vec{\mathbf{v}}$	$\nabla^2\phi$ (*resp.* $\nabla^2\vec{v}$)

Useful equations about Hamilton's operator ...

$$\nabla \cdot U\mathbf{A} = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$$

$$\nabla \times U\mathbf{A} = U\nabla \times \mathbf{A} - \mathbf{A} \times \nabla U$$

$$\nabla \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{A} \cdot \nabla \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \mathbf{A} \cdot \mathbf{B}\nabla - \mathbf{A} \cdot \nabla \mathbf{B}$$

$$\mathbf{B} \times (\nabla \times \mathbf{A}) = \mathbf{B} \cdot \mathbf{A}\nabla - \mathbf{B} \cdot \nabla \mathbf{A}$$

→ $\mathbf{A}$ is to be differentiated

$$\nabla \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}\nabla + \mathbf{A} \cdot \mathbf{B}\nabla$$
$$= \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$\mathbf{A} = \mathbf{B}$ →

$$\frac{1}{2}\nabla \mathbf{A}^2 = \mathbf{A} \cdot \nabla \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$$

$$\nabla \times \nabla \cdot \mathbf{A} = \nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla U = 0$$

→ valid when the order of differentiation is not important in the second mixed derivative